Dispersion of passive tracers in a velocity field with non-δ-correlated noise

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(Received 6 July 1998)

The diffusive properties in velocity fields whose small scales are parameterized by non-δ-correlated noise is investigated using multiscale technique. The analytical expression of the eddy diffusivity tensor is found for a two-dimensional (2D) steady shear flow and it is an increasing function of the characteristic noise decorrelation time τ. In order to study a generic flow $\boldsymbol{v}$, a small-τ expansion is performed and the first correction $O(\tau)$ to the effective diffusion coefficients is evaluated. This is done using two different approaches and it results that at the order τ the problem with a colored noise is equivalent to the δ-correlated case provided by a renormalization of the velocity field $\boldsymbol{v}\mapsto\tilde{\boldsymbol{v}}$ depending on τ. Two examples of 2D closed-streamlines velocity field are considered and in both the cases an enhancement of the diffusion is found. [S1063-651X(99)04204-X]

PACS number(s): 47.52.+j

I. INTRODUCTION

The problem of diffusion in a given velocity field has both theoretical and practical relevance in many different fields of science and engineering as, e.g., transport processes in chemical engineering and combustion studies [1]. The tracers transport, in particular the evolution of their concentration, plays an important role in many aspects of geophysics. For the oceanic flows, satellite data indicate that the mesoscale features, like eddies and cold filaments, advect temperature and nutrients over spatial and temporal scales longer than those of the geostrophic turbulence. The diffusion enhancement by a given velocity field has attracted a lot of works in the last years. In particular the role of the velocity field properties has been largely investigated while the effects of small-scale parameterization are not understood.

In this paper we will focus on the effects of a finite noise correlation time. This problem is relevant in studying the transport in the ocean since in this system the noise term comes from unresolved velocity scales which are correlated in time.

In Sec. II, by using the multiscale technique, we study the diffusion properties of the model proposed in Ref. [2] for transport in the upper mesoscale ocean. The transport is described by a Langevin equation with a Gaussian colored noise in time.

The aim is to understand whether a finite noise correlation time τ enhances or depresses the dispersion process in a given velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ with respect to the δ-correlated case $(\tau=0)$.

Exploiting the scale separation in the dynamics we derive, using the multiscale technique [3], an effective diffusive equation for the macrodynamics, the calculation of the effective diffusivity second-order tensor is reduced to the solution of one auxiliary partial differential equation [4–6].

In Sec. III we consider a shear flow, in this case the diffusion coefficient increases with τ. The solution of the auxiliary equation is, in general, quite difficult, therefore, to investigate the role of the finite τ in Sec. IV we perform a small-τ expansion. An alternative method is presented in the Appendix A.

In Sec. V we study the case of two closed-streamlines fields that mimics the transport in the Rayleigh-Bénard system: the quasi-two-dimensional flow studied by Shraiman in [7] and the AB flow. In both the cases the presence of a small correlation time enhances the diffusion process.

Conclusions are reserved for the final Sec. VI.

II. EFFECTIVE DIFFUSION EQUATION FOR GAUSSIAN COLORED NOISE

We consider large-scale and long-time dynamics of the model proposed in [8] and already studied in [2] for the transport of a fluid particle in the upper mesoscale ocean:

$$\frac{d}{dt}\boldsymbol{x}=\boldsymbol{v}(\boldsymbol{x},t)+\boldsymbol{s}(t), \tag{1}$$

where $\boldsymbol{v}$ is a d-dimensional incompressible velocity field $(\boldsymbol{\nabla}\cdot\boldsymbol{v}=0)$, for simplicity, periodic both in space and in time and $\boldsymbol{s}$ is a Gaussian random variable of zero mean and correlation function

$$\langle s_i(t)s_j(t')\rangle=\frac{\sigma^2}{\tau}\delta_{ij}e^{-[(|t-t'|)/\tau]}. \tag{2}$$

The term $\boldsymbol{v}(\boldsymbol{x},t)$ represents the part of the velocity field that one is able to resolve, i.e., the larger scale mean flow, whereas $\boldsymbol{s}(t)$ represents the part of the velocity field containing the subgridscale flow processes, e.g. the small-scale turbulence. The plausibility of such a description is discussed in [9–11]. In the limit $\tau\to 0$, resulting $e^{-|t-t'|/\tau}/\tau\to\delta(t-t')$, (2) reproduces the widely studied δ-correlated case

$$\langle s_i(t)s_j(t')\rangle=\sigma^2\delta_{ij}\delta(t-t'), \tag{3}$$

the diffusive properties of which we would like to compare with the τ-correlated noise case.

To study the dispersion of tracers evolving according to Eqs. (1) and (2) on large scales and long times we use the multiscale technique. This is a powerful mathematical method, also known as homogenization, for studying transport processes on time and spatial scales much larger than those of the velocity field $\boldsymbol{v}$. It has been already applied to the δ-correlated case [3] and it has been shown that the motion on large time and spatial scales is diffusive and it is described by an effective diffusion tensor which takes into account the effects of the advecting velocity on the bare diffusion coefficient σ^2.

To apply this method to the case of Gaussian colored noise, we first write Eqs. (1) and (2) into a Markovian process by enlarging the state space considering $\boldsymbol{s}(t)$ as a variable evolving according to the Langevin equation:

$$\frac{d}{dt}\boldsymbol{s}=-\frac{1}{\tau}\boldsymbol{s}+\boldsymbol{w}(t), \tag{4}$$

where now the noise $\boldsymbol{w}(t)$ is a white noise with correlation functions

$$\langle w_i(t)w_j(t')\rangle=2\left(\frac{\sigma}{\tau}\right)^2\delta_{ij}\delta(t-t'). \tag{5}$$

We have now a two-variable $(\boldsymbol{x},\boldsymbol{s})$ Markovian process whose associated Fokker-Planck equation can be easily obtained. Indeed, introducing

$$\boldsymbol{Y}=\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{s}\end{pmatrix};\quad \boldsymbol{W}=\begin{pmatrix}\boldsymbol{w}\\ \boldsymbol{w}\end{pmatrix};\quad \boldsymbol{V}=\begin{pmatrix}\boldsymbol{v}+\boldsymbol{s}\\ -\frac{1}{\tau}\boldsymbol{s}\end{pmatrix};\quad \hat{A}=\begin{pmatrix}\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{1}\end{pmatrix} \tag{6}$$

Eqs. (1) and (4) become

$$\frac{d}{dt}\boldsymbol{Y}=\boldsymbol{V}(\boldsymbol{Y},t)+\hat{A}\cdot\boldsymbol{W}(t). \tag{7}$$

The associated Fokker-Planck equation is

$$\partial_t\Theta=\left(\frac{\sigma}{\tau}\right)^2\boldsymbol{\partial}_{ss}^2\Theta-(\boldsymbol{v}+\boldsymbol{s})\cdot\boldsymbol{\partial}_x\Theta+\frac{1}{\tau}\boldsymbol{\partial}_s\cdot(\boldsymbol{s}\Theta), \tag{8}$$

where $\Theta=\Theta(\boldsymbol{x},\boldsymbol{s},t)$ denotes the probability density.

The doubling of the space dimension is the price to pay for having a Fokker-Planck equation. In Appendix A we discuss a different approach to the problem that does not double the dimension of the space, but leads in general to a non-Markovian master equation.

We can now apply the multiscale technique. Following [6] in addition to the *fast* variables $\boldsymbol{x}$ and t we introduce the *slow* variables defined as $\boldsymbol{X}=\epsilon\boldsymbol{x}$ and $T=\epsilon^2 t$ where $\epsilon\ll 1$ is the parameter controlling the separation between the small scales related to the velocity field $\boldsymbol{v}$ and the large scale related to the Θ variation. The two sets of variables are considered to be independent and so we have to make the substitution

$$\boldsymbol{\partial}_x\mapsto\boldsymbol{\partial}_x+\epsilon\boldsymbol{\partial}_X;\quad \partial_t\mapsto\partial_t+\epsilon^2\partial_T. \tag{9}$$

The solution of the Fokker-Planck equation (8) is sought as a perturbative series

$$\Theta(\boldsymbol{x},t,\boldsymbol{X},T,\boldsymbol{s})=\Theta^{(0)}+\epsilon\Theta^{(1)}+\epsilon^2\Theta^{(2)}+\cdots, \tag{10}$$

where the functions $\Theta^{(n)}$ depend on both fast and slow variables. By inserting Eqs. (9) and (10) into the Fokker-Planck equation (8), equating terms of equal powers in ϵ and choosing the solutions that have the same periodicities as the velocity field, we obtain a hierarchy of equations the first three of which are

$$\mathcal{D}\Theta^{(0)}=0, \tag{11}$$

$$\mathcal{D}\Theta^{(1)}=-(\boldsymbol{v}+\boldsymbol{s})\cdot\boldsymbol{\partial}_X\Theta^{(0)}, \tag{12}$$

$$\mathcal{D}\Theta^{(2)}=-(\boldsymbol{v}+\boldsymbol{s})\cdot\boldsymbol{\partial}_X\Theta^{(1)}-\partial_T\Theta^{(0)}, \tag{13}$$

where the operator $\mathcal{D}$ is defined as

$$\mathcal{D}=\partial_t+(\boldsymbol{v}+\boldsymbol{s})\cdot\boldsymbol{\partial}_x-\frac{1}{\tau}\boldsymbol{\partial}_s(\boldsymbol{s})-\left(\frac{\sigma}{\tau}\right)^2\boldsymbol{\partial}_{ss}{}^2. \tag{14}$$

In order to solve Eq. (11) we make use of scale separation and we write the solution $\Theta^{(0)}$ as the sum of two terms: $\langle\Theta^{(0)}\rangle(\boldsymbol{X},T,\boldsymbol{s})$ depending on the *slow* variables and $\tilde{\Theta}^{(0)}(\boldsymbol{x},t,\boldsymbol{s})$ depending on the *fast variables*. Here and in the following the $\langle\cdot\rangle$ indicates the average over the *fast* variables.

Equation (11) then splits into the two equations

$$\mathcal{D}\tilde{\Theta}^{(0)}=0, \tag{15}$$

$$\left(\frac{\sigma}{\tau}\right)^2\boldsymbol{\partial}_{ss}{}^2\langle\Theta^{(0)}\rangle+\frac{1}{\tau}\boldsymbol{\partial}_s(\boldsymbol{s}\langle\Theta^{(0)}\rangle)=0. \tag{16}$$

One can show [6] that the solution $\tilde{\Theta}^{(0)}$ will relax to a constant with respect to fast variables, so we can simply take

$$\Theta^{(0)}(\boldsymbol{x},t,\boldsymbol{X},T,\boldsymbol{s})=\langle\Theta^{(0)}\rangle(\boldsymbol{X},T,\boldsymbol{s}) \tag{17}$$

and write the solution of Eq. (16) as

$$\langle\Theta^{(0)}\rangle=Q(\boldsymbol{X},T)\mathcal{P}(s) \tag{18}$$

where $\mathcal{P}(s)$ is defined as

$$\mathcal{P}(s)=\frac{e^{-\tau s^2/2\sigma^2}}{(2\pi\sigma^2/\tau)^{d/2}} \tag{19}$$

with d the dimension of the $\boldsymbol{x}$ space.

By using Eq. (18) we see that Eq. (12) can be written as

$$\mathcal{D}\Theta^{(1)}=\boldsymbol{f}(\boldsymbol{x},t,\boldsymbol{s})\cdot\boldsymbol{G}(\boldsymbol{X},T) \tag{20}$$

with

$$\boldsymbol{f}(\boldsymbol{x},t,\boldsymbol{s})=-(\boldsymbol{v}+\boldsymbol{s})\mathcal{P}(s),\quad \boldsymbol{G}(\boldsymbol{X},T)=\boldsymbol{\partial}_X Q(\boldsymbol{X},T) \tag{21}$$

with solution

$$\Theta^{(1)}(\boldsymbol{x},t,\boldsymbol{X},T,\boldsymbol{s})=\boldsymbol{\chi}(\boldsymbol{x},t,\boldsymbol{s})\cdot\boldsymbol{G}(\boldsymbol{X},T). \tag{22}$$

The vector field $\boldsymbol{\chi}$ is called the auxiliary field and it solves the auxiliary equation

$$\mathcal{D}\boldsymbol{\chi}=\boldsymbol{f}. \tag{23}$$

Finally by averaging Eq. (13) over the fast variables, $\langle\cdot\rangle$, and integrating over $\boldsymbol{s}$, $\overline{(\cdot)}$, we obtain

$$\boldsymbol{\partial}_X\overline{\langle(\boldsymbol{v}+\boldsymbol{s})\Theta^{(1)}\rangle}=-\partial_T\overline{\langle\Theta^{(0)}\rangle}=-\partial_T Q(\boldsymbol{X},T), \tag{24}$$

which, using Eq. (22), becomes

$$\partial_T Q=-\overline{\langle(v_j+s_j)\chi_i\rangle}\partial^2_{X_iX_j}Q=D^E_{ij}\partial^2_{X_iX_j}Q. \tag{25}$$

This is the diffusion equation describing the large-scale dynamics, i.e., the dynamics in the slow variables. The effective eddy diffusivity tensor D^E_{ij} is given by

$$D^E_{ij}=-\tfrac{1}{2}[\overline{\langle(v_i+s_i)\chi_j\rangle}+\overline{\langle(v_j+s_j)\chi_i\rangle}]. \tag{26}$$

From the auxiliary equation (23) one can show that the D^E_{ij} is positive definite. Indeed, if we consider the ith and the jth component of Eq. (23), multiply by χ_j and χ_i, respectively, sum the two terms and average the result over the periodicities, $\langle\cdot\rangle$, and integrate over the random variable $\boldsymbol{s}$ we end up with

$$D^E_{ij}=\left(\frac{\sigma}{\tau}\right)^2\int d^ds\,\partial_s\langle\chi_i\rangle\partial_s\langle\chi_j\rangle\mathcal{P}(s)\geqslant 0. \tag{27}$$

This result can be extended to non-periodic velocity field following the prescriptions in [12].

III. A SOLVABLE CASE: THE STATIONARY SHEAR FLOW

The resolution of the auxiliary field equation for a generic $\boldsymbol{v}$ is not an easy task. Therefore not trivial solvable cases are useful to understand the properties of the solution. In particular the auxiliary equation can be resolved for parallel flows, which in two dimensions have the form

$$\boldsymbol{v}(x,y)=(v(y),0), \tag{28}$$

where $v(y)$ is an arbitrary function of y. Note that these flows automatically satisfy $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$. To evaluate the effective diffusion coefficients we first write the solution of the auxiliary equation as

$$\chi_i(\boldsymbol{x},t,\boldsymbol{s},\boldsymbol{x}',t',\boldsymbol{s}')=-\int dt'\,d\boldsymbol{x}'\,d\boldsymbol{s}'\,\mathcal{G}(\boldsymbol{x},t,\boldsymbol{s},\boldsymbol{x}',t',\boldsymbol{s}')\times(v_i(\boldsymbol{x}')+s_i')\mathcal{P}(s') \tag{29}$$

where $\mathcal{G}$ is the Green function of the operator $\mathcal{D}$:

$$\mathcal{D}\mathcal{G}=\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t')\,\delta(\boldsymbol{s}-\boldsymbol{s}'). \tag{30}$$

Inserting Eq. (29) into Eq. (26) we have

$$D^E_{ij}=-\int dt\,d\boldsymbol{x}\,d\boldsymbol{s}\,dt'\,d\boldsymbol{x}'\,d\boldsymbol{s}'[v_i(\boldsymbol{x})+s_i]\mathcal{G}(\boldsymbol{x},t,\boldsymbol{s},\boldsymbol{x}',t',\boldsymbol{s}')\times[v_j(\boldsymbol{x}')+s_j']\mathcal{P}(s'). \tag{31}$$

Now we note that the Green function $\mathcal{G}$ can be written as

$$\mathcal{G}=\langle\,\delta(\boldsymbol{x}-\boldsymbol{x}(t;\boldsymbol{x}',t'))\,\delta(\boldsymbol{s}-\boldsymbol{s}(t;\boldsymbol{s}',t'))\rangle_w, \tag{32}$$

where the average is over the realizations of the white noise $\boldsymbol{w}$, $\boldsymbol{x}(t;\boldsymbol{x}',t')$ and $\boldsymbol{s}(t;\boldsymbol{s}',t')$ are the solutions of Eqs. (1), (4), and (28) with initial condition $\boldsymbol{x}'=\boldsymbol{x}(t';\boldsymbol{x}',t')$ and $\boldsymbol{s}'=\boldsymbol{s}(t';\boldsymbol{s}',t')$. For the velocity field (28) the solutions of Eqs. (1) and (4) can be written as

$$x(t)=x'+\int_{t'}^{t}dt_1 v(y(t_1))+\int_{t'}^{t}dt_1 s_1(t_1),$$

$$y(t)=y'+\int_{t'}^{t}dt_1 s_2(t_1), \tag{33}$$

$$\boldsymbol{s}(t)=\boldsymbol{s}'e^{-(t-t')/\tau}+\int_{t'}^{t}dt_1\boldsymbol{w}(t_1)e^{-(t-t_1)/\tau}.$$

Inserting Eqs. (32) and (33) into Eq. (31), after some straightforward algebra one obtains

$$D^E_{11}=\sigma^2+\frac{1}{2\pi}\int dk|\hat{v}(k)|^2\lim_{t\to\infty}\int_0^t dt'\times\exp\{-\sigma^2k^2[(t-t')-\tau(1-e^{-[(t-t')/\tau]})]\} \tag{34}$$

and

$$D^E_{12}=0,\quad D^E_{22}=\sigma^2, \tag{35}$$

where $\hat{v}(k)$ is the Fourier transform of $v(y)$. The same result can be obtained directly from the definition

$$D^E_{ij}=\lim_{t\to\infty}\frac{1}{2t}\langle[x_i(t)-\langle x_i\rangle][x_j(t)-\langle x_j\rangle]\rangle \tag{36}$$

using Eq. (33). Now, because of the inequality

$$\exp[\sigma^2k^2\tau(1-e^{-[(t-t')/\tau]})]\geqslant 1 \tag{37}$$

one has

$$D^E_{11}(\tau)\geqslant\sigma^2+\frac{1}{2\pi}\int dk|\hat{v}(k)|^2\lim_{t\to\infty}\int_0^t dt'\,e^{-\sigma^2k^2(t-t')}=\sigma^2+\frac{1}{2\pi}\int dk\frac{|\hat{v}(k)|^2}{\sigma^2k^2}=D^E_{11}(0). \tag{38}$$

Therefore for a stationary parallel flow a colored noise produces an enhancement of the dispersion. Similar equations can be obtained for a time-dependent shear flow. However, in this case it is not simple to see the sign of the correction. The results will be reported elsewhere. It is trivial to show that the results in this section hold also for a $3d$ shear flow:

$$\mathbf{v}(x,y,z)=(v(y,z),0,0). \tag{39}$$

IV. EDDY DIFFUSIVITY FOR SHORT NOISE CORRELATION TIME

By using multiscale technique, the calculation of the eddy diffusivities has been reduced to the solution of the auxiliary equation (23). Numerical methods are generally needed to solve it but to do so we have to work in a $2d$-dimensional space. In general, this is not feasible, so, to get more insight the generic $\boldsymbol{v}$ case we study the small τ case and expand the auxiliary field χ in a power series of τ. A typical time of the physical system that can be compared to τ is $\tau_s = \lambda/\langle v^2\rangle^{1/2}$, i.e., the average time it takes a particle to travel a characteristic length λ.

Taking

$$\alpha_i = \sqrt{\tau} s_i,$$

$$\chi_i = \frac{\tau}{2\pi\sigma^2}\sum_{k=0}^{\infty} \chi_i^{(k)}(\boldsymbol{x},\boldsymbol{\alpha},t)\,\tau^{k/2}$$

we obtain the following expression for the eddy diffusivity tensor:

$$D_{ij}^E = -\frac{1}{2}\frac{1}{2\pi\sigma^2}\Bigg[\sum_k \tau^{k/2}(\langle v_i\overline{\chi_j^{(k)}}\rangle + \langle v_j\overline{\chi_i^{(k)}}\rangle) + \tau^{[(k-1)/2]} \times(\langle\overline{\alpha_i\chi_j^{(k)}}\rangle + \langle\overline{\alpha_j\chi_i^{(k)}}\rangle)\Bigg] \tag{40}$$

and for the auxiliary equation

$$\sum_k [\tau^{k/2}(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\partial}_x) + \tau^{[(k-1)/2]}\boldsymbol{\alpha}\cdot\boldsymbol{\partial}_x - \tau^{[(k-2)/2]} \times(\boldsymbol{\partial}_\alpha(\boldsymbol{\alpha}) - \sigma^2\partial^2_{\alpha\alpha})]\chi_i^{(k)} = -\left(v_i + \frac{\alpha_i}{\sqrt{\tau}}\right)e^{-(\alpha^2/2\sigma^2)}. \tag{41}$$

From the expression (40) we see that in order to determine the correction of order τ to D_{ij}^E, we need the quantities $\langle\chi_i^{(0)}\rangle$, $\overline{\chi_i^{(0)}}$, $\langle\chi_i^{(1)}\rangle$, $\overline{\chi_i^{(1)}}$, $\langle\chi_i^{(2)}\rangle$, $\overline{\chi_i^{(2)}}$, and $\langle\chi_i^{(3)}\rangle$. The fields $\chi_i^{(k)}$ obey the equations

$$\mathcal{O}_\alpha\chi_i^{(0)} = 0, \tag{42}$$

$$\mathcal{O}_\alpha\chi_i^{(1)} = \boldsymbol{\alpha}\cdot\boldsymbol{\partial}_x\chi_i^{(0)} + \alpha_i e^{-(\alpha^2/2\sigma^2)}, \tag{43}$$

$$\mathcal{O}_\alpha\chi_i^{(2)} = (\partial_t + \boldsymbol{v}\cdot\boldsymbol{\partial}_x)\chi_i^{(0)} + \boldsymbol{\alpha}\cdot\boldsymbol{\partial}_x\chi_i^{(1)} + v_i e^{-(\alpha^2/2\sigma^2)}, \tag{44}$$

$$\mathcal{O}_\alpha\chi_i^{(h)} = (\partial_t + \boldsymbol{v}\cdot\boldsymbol{\partial}_x)\chi_i^{(h-2)} + \boldsymbol{\alpha}\cdot\boldsymbol{\partial}_x\chi_i^{(h-1)}, \quad h\geqslant 3, \tag{45}$$

where the operator $\mathcal{O}_\alpha$ is defined as

$$\mathcal{O}_\alpha = \boldsymbol{\partial}_\alpha\cdot(\boldsymbol{\alpha}) - \sigma^2\partial^2_{\alpha\alpha}. \tag{46}$$

The solutions of Eqs. (42), (43), and (44) can be written in the form

$$\chi_i^{(0)} = \tilde{\chi}_i^{(0)}(\boldsymbol{x},t)e^{-(\alpha^2/2\sigma^2)}, \tag{47}$$

$$\chi_i^{(1)} = (a_{1i}\alpha_1 + a_{2i}\alpha_2 + a_{3i})e^{-(\alpha^2/2\sigma^2)}, \tag{48}$$

$$\chi_i^{(2)} = (b_{1i}\alpha_1^2 + b_{2i}\alpha_2^2 + b_{3i}\alpha_1\alpha_2 + b_{4i}\alpha_1 + b_{5i}\alpha_2 + b_{6i})e^{-(\alpha^2/2\sigma^2)}. \tag{49}$$

The coefficients a_{ij} and b_{ij} are functions of $\boldsymbol{x}$ and t while for the field $\chi_i^{(3)}$

$$\langle\chi_i^{(3)}\rangle = c(\boldsymbol{x},t)e^{-(\alpha^2/2\sigma^2)}. \tag{50}$$

By inserting expressions (47), (48), and (49) into Eqs. (42), (43), (44), and (45) we can determine the coefficients a_{ij} for $i=1,2$ and b_{ij} for $i=1,\ldots,5$ and then by integrating over α the equations for $\chi_j^{(2)}$, $\chi_j^{(3)}$, and $\chi_j^{(4)}$, respectively, we finally have the equations for the remaining functions $\tilde{\chi}_i^{(0)}$, a_{3i}, and b_{6i},

$$\mathcal{O}_{xt}\tilde{\chi}_i^{(0)} = -v_i, \tag{51}$$

$$\mathcal{O}_{xt}a_{3i} = 0, \tag{52}$$

$$\mathcal{O}_{xt}b_{6i} = \tfrac{3}{2}\sigma^2[\partial^2 v_i + \partial^2 v_j\partial_j\tilde{\chi}_i^{(0)}] + 2\sigma^2\partial_j v_m\partial^2_{jm}\tilde{\chi}_i^{(0)}, \tag{53}$$

where

$$\mathcal{O}_{xt} = \partial_t + \boldsymbol{v}\cdot\boldsymbol{\partial} - \sigma^2\partial^2. \tag{54}$$

The D_{ij}^E coefficients at the first order in τ read

$$D_{ij}^E = \sigma^2\delta_{ij} - \tfrac{1}{2}(\langle v_i\tilde{\chi}_j^{(0)}\rangle + \langle v_j\tilde{\chi}_i^{(0)}\rangle) - \frac{\tau}{2}\Bigg[\frac{\sigma^2}{2}(\langle v_i\partial^2\tilde{\chi}_j^{(0)}\rangle + \langle v_j\partial^2\tilde{\chi}_i^{(0)}\rangle) + \langle v_i b_{6j}\rangle + \langle v_j b_{6i}\rangle\Bigg]. \tag{55}$$

We note that instead of the $2d$-dimensional equation (26) we have now a system of two d-dimensional equations (51) and (53) without the random variable $\boldsymbol{s}$. Of course this is numerically much more convenient.

We note that by defining new velocity and auxiliary fields as

$$\tilde{\boldsymbol{v}} = \boldsymbol{v} - \frac{\sigma^2\tau}{2}\partial^2\boldsymbol{v}, \tag{56}$$

$$\tilde{\boldsymbol{\chi}} = \tilde{\boldsymbol{\chi}}^{(0)} + \tau(\boldsymbol{b}_6 + \sigma^2\partial^2\tilde{\boldsymbol{\chi}}^{(0)}) \tag{57}$$

and negletting terms of $O(\tau^2)$ equations (51), (53), and (55) can be written as

$$(\partial_t + \tilde{\boldsymbol{v}}\boldsymbol{\partial} - \sigma^2\partial^2)\tilde{\chi}_i = \tilde{v}_i \tag{58}$$

and

$$D_{ij}^E = \sigma^2\delta_{ij} - \tfrac{1}{2}(\langle\tilde{v}_i\tilde{\chi}_j\rangle + \langle\tilde{v}_j\tilde{\chi}_i\rangle), \tag{59}$$

formally equivalent to the Gaussian white noise result.

In Appendix A we use a different method to obtain the expression of the eddy diffusivity tensor for small τ up to $O(\tau^2)$. By starting from the Master equation associated with the Langevin equation (1) and using a small τ expansion we derive the Fokker-Planck equation

$$\partial_t\Theta(\mathbf{x},t) = -\partial_i[v_i(\mathbf{x},t)\Theta(\mathbf{x},t)] + \partial^2_{ij}[\mathcal{D}_{ij}(\mathbf{x},t)\Theta(\mathbf{x},t)] \quad (60)$$

with

$$\mathcal{D}_{ij}(\mathbf{x},t) = \sigma^2\left[\delta_{ij} + \frac{\tau}{2}[\partial_i v_j(\mathbf{x},t) + \partial_j v_i(\mathbf{x},t)]\right]. \quad (61)$$

Now, by applying multiscale technique to Eq. (60) we end up with the following equations:

$$\mathcal{O}_{xt} w_i^{(0)} = -v_i, \quad (62)$$

$$\mathcal{O}_{xt} w_i^{(1)} = \sigma^2(\partial^2 v_i + \partial^2 v_j \partial_j w_i^{(0)} + \partial_k v_j \partial^2_{kj} w_i^{(0)}), \quad (63)$$

$$D^E_{ij} = \sigma^2\delta_{ij} - \tfrac{1}{2}(\langle v_i w_j^{(0)}\rangle + \langle v_j w_i^{(0)}\rangle) - \frac{\tau}{2}[(\langle v_i \partial^2 w_j^{(0)}\rangle + \langle v_j \partial^2 w_i^{(0)}\rangle) + \langle v_i w_j^{(1)}\rangle + \langle v_j w_i^{(1)}\rangle], \quad (64)$$

which differ from the previous ones.

This result raises the question about the validity of the two sets of equations and hence of the expansions. In order to answer to this question we firstly note that the two sets can be considered as a particular case of the *generalized* equations

$$\mathcal{O}_{xt} w_i^{(0)} = -v_i, \quad (65)$$

$$\mathcal{O}_{xt} w_i^{(1)} = \sigma^2((p-q)\partial^2 v_i + (p-q)\partial^2 v_j \partial_j w_i^{(0)} + p\,\partial_k v_j \partial^2_{kj} w_i^{(0)}) \quad (66)$$

$$D^E_{ij} = \sigma^2\delta_{ij} - \tfrac{1}{2}(\langle v_i w_j^{(0)}\rangle + \langle v_j w_i^{(0)}\rangle) - \frac{\tau}{2}[q\sigma^2(\langle v_i \partial^2 w_j^{(0)}\rangle + \langle v_j \partial^2 w_i^{(0)}\rangle) + \langle v_i w_j^{(1)}\rangle + \langle v_j w_i^{(1)}\rangle] \quad (67)$$

obtained starting from the master equation

$$\partial_t\Theta(\mathbf{x},t) = -\tilde{v}_i(\mathbf{x},t)\partial_i\Theta(\mathbf{x},t) + \partial^2_{ij}[\mathcal{D}_{ij}(\mathbf{x},t)\Theta(\mathbf{x},t)], \quad (68)$$

where

$$\tilde{v}_i = v_i + q\sigma^2\tau\partial^2 v_i, \quad (69)$$

$$\mathcal{D}_{ij} = \sigma^2\left(\delta_{ij} + \frac{p\tau}{2}(\partial_i v_j + \partial_j v_i)\right) \quad (70)$$

and by applying the multiscale technique.

The role of the two free parameters p and q is of generalizing the master equation (60) being our feeling that there exists a family of different microscopic process specified by $\tilde{\mathbf{v}}$ and $\mathcal{D}_{ij}$ that correspond to the same macroscopic diffusive process specified by D^E_{ij}.

The first set consisting of the equations (51), (53), and (55) corresponds to $q=1/2$ and $p=2$, whereas the second one consisting of the equations (62), (63), and (64) corresponds to $q=0$ and $p=1$. Now a closer analysis of the diffusion coefficients (70) reveals that it may take negative values, introducing unphysical singularities into the problem. Moreover the multiscale technique requires a D^E_{ij} definite positive. This means that Eq. (68) does not constitute a genuine Markovian process with a well defined corresponding Langevin equation driven by a Gaussian white noise. In general this expansion do not converge uniformly in $\mathbf{x}$ and the range of validity is restricted to $\tau \ll 1$, $\tau/\sigma^2 \ll 1$ (in dimensionless unit), thus they are asymptotic estimate for $\tau \to 0$. In this range of equations validity we do not have any *a priori* arguments for choosing one of the two sets of equations and it is for this reason that we expect to obtain the same value of the eddy diffusivity tensor using the two sets. This is indeed the case for the parallel shear flows (see Appendix B). This result can be thought as a first indication of the equivalence (with respect to the value of the diffusivity) of the *generalized* equations.

In general we do not expect that all possible choices of q and p give the same D^E_{ij} but we can think to select the class of equivalent process by applying the *generalized* equations to the parallel shear flow and imposing D^E_{ij} equal to the known expression (38) independent on both q and p. This calculation is reported in Appendix B and it ends up in the following condition:

$$p = 2q + 1. \quad (71)$$

All these considerations suggest that for short τ there exists a class of equivalent equations depending on the parameter p that lead to the same eddy diffusivity tensor.

If this is the case, among the all possible choices of q and p consistent with Eq. (71), we can choose: $p=0$ and $q=-1/2$. In this case $\mathcal{D}_{ij}=\sigma^2\delta_{ij}$ and Eq. (60) reduces to a truly Markovian process with the associated Langevin equation,

$$\frac{d}{dt}\mathbf{x} = \tilde{\mathbf{v}}(\mathbf{x},t) + \boldsymbol{\xi}(t), \quad (72)$$

where $\boldsymbol{\xi}$ is a Gaussian white noise. This is the microscopic Markovian process that approximates the long time and large space transport properties of a colored noise process with short noise correlation time. In other words, to study the diffusion properties of Eqs. (1)–(2) for small τ we can replace the original colored noise process with the process described by Eq. (72). This will give the correct diffusion coefficients up to $O(\tau^2)$.

We have checked numerically that for the AB flow the eddy diffusivity tensor assumes the same value for the three different choices of the parameters q and p consistent with Eq. (71): $p=2$ and $q=1/2$, $p=1$ and $q=0$, $p=0$ and $q=-1/2$. This gives us confidence in our conclusions.

V. FLOWS WITH CLOSED STREAMLINES

We apply now our analysis to two models for the Rayleigh-Bénard steady convection: the first one consists of an horizontal extent of convection cells much larger than its height so that the flow can be considered quasi-two-dimensional; the second one is the two-dimensional AB flow made of a structure periodically repeated in the space.

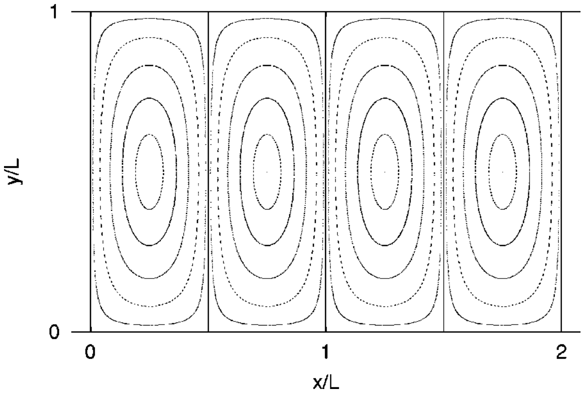

FIG. 1. The streamlines for the *quasi*-two-dimensional flow (73) with $k=2$.

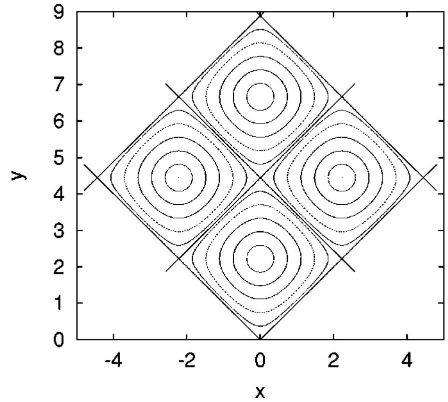

FIG. 2. The streamlines for the two-dimensional AB flow (78) with $A=B=1$.

‘A. A quasi-two-dimensional flow

We consider the flow discussed by Shraiman [7]. This is described by the stream function

$$\psi(x,y)=\frac{vL}{\pi k}\sin\left(\frac{\pi k}{L}x\right)\sin\left(\frac{\pi}{L}y\right) \tag{73}$$

with v being the characteristic velocity, L the height of the cell ($y\in[0,L]$), and L/k the x-periodicity of the roll pattern. The top and the bottom plates of the cell are assumed impermeable for the passive scalar so that the appropriate boundary conditions for the tracers density function Θ are $\partial_y\Theta|_{y=0,L}=0$. The streamlines of the flow are illustrated in Fig. 1. Using Fokker-Planck equation

$$\partial_t\Theta=-\boldsymbol{v}\cdot\boldsymbol{\partial}\Theta+\sigma^2\partial^2\Theta, \tag{74}$$

for large Péclet number $\mathrm{Pe}=vL/\sigma^2$, $k=2$ and $\tau=0$ the eddy diffusivity coefficient $D_{11}^E(0)$ has been calculated in [7] and it is

$$D_{11}^E(0)=\frac{\sigma^2}{\sqrt{\pi}}\sqrt{\mathrm{Pe}}=\sqrt{\frac{vL\sigma^2}{\pi}}. \tag{75}$$

According to the results of the previous section, the colored noise case with small τ is described up to order $O(\tau^2)$ by the same Fokker-Planck equation provided the velocity field is renormalized as Eq. (69) with $q=-1/2$. Taking into account that $v_x=-\partial_y\psi$, $v_y=\partial_x\psi$, and using Eq. (73), we have

$$\partial^2\psi=-(\pi/L)^2(1+k^2)\psi.$$

Therefore up to order $\mathcal{O}(\tau^2)$ the diffusion is described by

$$\tilde{\psi}(x,y)=\left[1+\frac{\sigma^2\tau}{2}\left(\frac{\pi}{L}\right)^2(1+k^2)\right]\psi(x,y)$$
$$=c(\sigma^2,\tau,k)\psi(x,y). \tag{76}$$

Since c does not depend on $\boldsymbol{x}$ and t, we can repeat the calculation of Shraiman and obtain under the same conditions of Eq. (75) the expression for the eddy diffusivity coefficient

$$D_{11}^E(\tau)=\frac{\sigma^2}{\sqrt{\pi}}\sqrt{c\,\mathrm{Pe}}=D_{11}^E(0)\sqrt{1+\frac{5\sigma^2\tau}{2}\left(\frac{\pi}{L}\right)^2}$$
$$=D_{11}^E(0)\left[1+\frac{5\sigma^2\tau}{4}\left(\frac{\pi}{L}\right)^2\right]+O(\tau^2). \tag{77}$$

We then conclude that, in this case, a *small* τ enhances the diffusion coefficient.

The same result can be deduced from the multiscale equations (58) and (59): in fact because of the structure of these two equations it is not difficult to show that if we change only the module of the velocity field ($\tilde{\boldsymbol{v}}=c\boldsymbol{v}$) and we know the explicit form of $D_{ij}^E=f(\boldsymbol{v})$ as a function of $\boldsymbol{v}$ we have $\tilde{D}_{ij}^E=f(\tilde{\boldsymbol{v}})$. For large Péclet number and $\tau=0$ the function $f(\boldsymbol{v})$ is given by Eq. (75) from which Eq. (77) follows.

B. *AB* flow

The AB flow is given by the velocity field

$$v(x,y)=(B\cos(y),A\cos(x)). \tag{78}$$

For $A=B=1$ the streamlines form a closed periodically repeated structure made of four cells as shown in Fig. 2. We expect that the diffusive behavior of such a system is similar to the previous case, in fact we know [13] that for small Péclet number Pe the eddy diffusivity tensor is proportional to $\sqrt{\mathrm{Pe}}$ like in the quasi-two-dimensional case.

In Fig. 3 the behavior of the quantity $\Delta=[D^E(\tau)-D^E(0)]/[D^E(0)\tau]$ versus σ^2 is shown. The correction Δ has been calculated by integrating numerically the equations (51) and (53) and evaluating the quantity

$$-\frac{1}{D^E(0)}\left[\frac{\sigma^2}{2}\langle v_1\partial^2\tilde{\chi}_1^{(0)}\rangle+\langle v_1 b_{61}\rangle\right]=\frac{D^E(\tau)-D^E(0)}{D^E(0)\tau}=\Delta \tag{79}$$

for different values of σ^2. The equations are solved by using a pseudospectral method [14] in the basic periodicity cell with a grid mesh of 64×64 points. Dealiasing has been obtained by a proper circular truncation which ensures better isotropy of numerical treatment.

It is evident that also in this case the numerical results follow a linear behavior with a positive angular coefficient

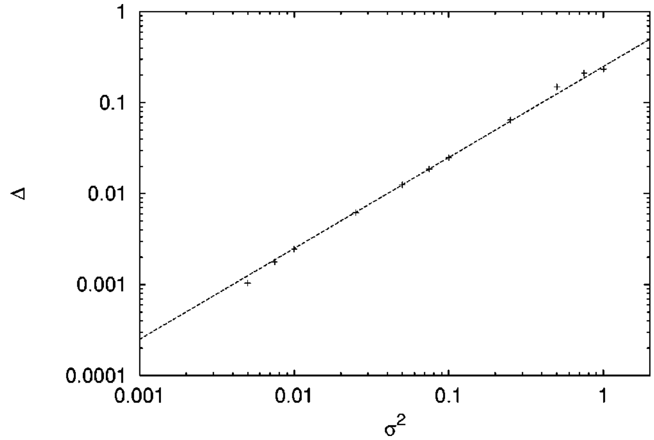

FIG. 3. The ratio Δ as a function of the σ^2 for the two-dimensional AB flow with $A=B=1$. The continuous line is the prediction obtained by Eq. (81) while the points are the numerical results obtained from Eq. (79). All quantities here are assumed to be dimensionless.

and so we can conclude that the introduction of the colored noise leads to an enhancement of the diffusion. In particular, we can see that the numerical results follow very well the line $\Delta=\sigma^2/4$. This is not surprising because we know that for large Pe

$$D^E(0)=D^E_{11}(0)=D^E_{22}(0)=C_1\sqrt{\mathrm{Pe}}=C_2\sqrt{v}; \tag{80}$$

therefore using the same arguments of the previous section we can deduce that

$$D^E(\tau)=D^E(0)\sqrt{1+\tfrac{1}{2}\sigma^2\tau}=D^E(0)(1+\tfrac{1}{4}\sigma^2\tau+O(\tau^2)), \tag{81}$$

in a very good agreement with the numerical results.

VI. SUMMARY AND CONCLUSIONS

In this paper we have studied the transport properties in velocity fields whose small scales are parameterized by Gaussian colored noise. We analyzed, in particular, the effects of a finite noise correlation time τ on the diffusive properties for large time and spatial scales. In this limit, using the multiscale technique, we derive the diffusion equation (25) and the associated effective diffusion tensor. The latter is obtained, once the velocity field $\boldsymbol{v}$ is given, by the solution of an auxiliary equation, see Eqs. (23) and (26), of the same structure of the original Fokker-Planck equation [15]. The former is, however, an exact result for the diffusive regime valid for very long times, thus avoiding all finite time effects of the Fokker-Planck equation or the associated Langevin equation (1),(2).

The auxiliary equation cannot be solved for a generic velocity field, nevertheless there are nontrivial flows for which the solution can be found. This is the case for the steady parallel flow for which the effective diffusion coefficient is an increasing function of τ. To study in more details the effects of a small noise correlation time for a generic $\boldsymbol{v}$ we have performed a small-τ expansion and evaluated the first correction $O(\tau)$ to the effective diffusion coefficient. This is done by using two different approach. We find that to order $O(\tau)$ there exist a one-parameter family of flows with the same diffusion properties. This invariance can be used to pick up the most convenient microscopic dynamics, from both analytical and numerical porpoises. We apply the small-τ results to two two–dimensional model flows with closed streamlines. In both the cases we find an enhancement of the diffusion.

The enhancement of the diffusion for small τ has been interpreted in [16] in terms of interference mechanism between turbulent and molecular diffusion. The colored noise makes the diffusion particles forgotten of their previous positions less rapidly than in the white-noise case, thus the Lagrangian correlation time increases and so does the eddy-diffusivity. The study of the problem for τ not small is the object of current work.

ACKNOWLEDGMENTS

We warmly thank Angelo Vulpiani, Massimo Vergassola, and Andrea Mazzino for extensive, stimulating, and very useful discussions and suggestions. P.C. is grateful to the European Science Foundation for a TAO exchange grant. This work has been partially supported by the INFM (Progetto Ricerca Avanzata–TURBO) and by MURST (Program No. 9702265437).

APPENDIX A: MASTER EQUATION FOR COLORED NOISE AND SMALL τ

In this appendix we derive the master equation for the probability density $\Theta(\boldsymbol{x},t)$ in the limit of small τ for the process described by the Langevin equation (1) and (2). If the random variable $\boldsymbol{s}$ is a Gaussian white noise of zero mean, $\Theta(\boldsymbol{x},t)$ satisfies the Fokker-Planck equation. We address here the case of a colored noise. Now the process $\boldsymbol{x}$ is non-Markovian and no exact simple equation for Θ is known. Let us consider an Ornstein-Uhlenbeck process $\mathbf{s}$, that is a zero mean Gaussian process with correlations [c.f.r Eq. (2)]

$$C_{ij}(t,t')=\langle s_i(t)s_j(t')\rangle=\frac{\sigma^2}{\tau}\delta_{ij}e^{-(|t-t'|)/\tau}, \tag{A1}$$

where τ is the correlation time. The probability density is given by

$$\Theta(\boldsymbol{x},t)=\langle\delta(\boldsymbol{x}(t)-\boldsymbol{x})\rangle, \tag{A2}$$

where $\boldsymbol{x}(t)$ is a solution of Eq. (1) for a given realization of $\boldsymbol{s}$ and for a given initial condition. The average is taken over the noise realizations. Taking the time derivative of Eq. (A2) and using Eq. (1) one gets

$$\partial_t\Theta(\boldsymbol{x},t)=-\partial_i[v_i(\boldsymbol{x},t)\Theta(\boldsymbol{x},t)]+\partial_i\langle s_i(t)\delta(\boldsymbol{x}(t)-\boldsymbol{x})\rangle. \tag{A3}$$

Taking advantage of the Gaussian nature of $\boldsymbol{s}$ the average in Eq. (A3) can be rewritten as

$$\partial_t \Theta(\boldsymbol{x},t) = -\partial_i[v_i(\boldsymbol{x},t)\Theta(\boldsymbol{x},t)] + \partial_{ij}^2 \int_{t_0}^{t} dt' \left(\frac{\sigma^2}{\tau}\right) \times e^{-[(t-t')/\tau]} \left\langle \frac{\delta x_j(t)}{\delta s_i(t')} \delta(\boldsymbol{x}(t)-\boldsymbol{x}) \right\rangle. \quad \text{(A4)}$$

Because of the δ function a closed equation is possible only if the functional derivative either does not involve the process $\boldsymbol{x}$ or depends on it solely at the ''Markovian'' end point $t=t'$. At this stage thus we cannot simplify the master equation any further. In the limit of small correlation time τ a closed equation can be derived by performing the largely used small-τ expansion. If the noise is close to the white noise limit ($\tau=0$) it is reasonable to expand the functional derivative about its Markovian value, i.e., the one obtained for the δ-correlated noise. The Taylor expansion of $\delta x_j(t)/\delta s_i(t')$ around the Markovian end point $t'=t$ is

$$\frac{\delta x_j(t)}{\delta s_i(t')} = \left.\frac{\delta x_j(t)}{\delta s_i(t')}\right|_{t'=t} + \frac{d}{dt'} \left.\frac{\delta x_j(t)}{\delta s_i(t')}\right|_{t'=t} (t'-t) + \cdots = \delta_{ij} - \partial_i v_j(\boldsymbol{x},t)(t'-t) + \cdots. \quad \text{(A5)}$$

Inserting the expansion (A5) into Eq. (A4), keeping only the first terms in τ and neglecting the transients, i.e., letting t_0 expand to $-\infty$, we obtain after straightforward algebra the small τ master equation:

$$\partial_t \Theta(\boldsymbol{x},t) = -\partial_i[v_i(\boldsymbol{x},t)\Theta(\boldsymbol{x},t)] + \partial_{ij}^2[\mathcal{D}_{ij}(\boldsymbol{x},t)\Theta(\boldsymbol{x},t)], \quad \text{(A6)}$$

where

$$\mathcal{D}_{ij}(\boldsymbol{x},t) = \sigma^2 \left[\delta_{ij} + \frac{\tau}{2}[\partial_i v_j(\boldsymbol{x},t) + \partial_j v_i(\boldsymbol{x},t)] \right] \quad \text{(A7)}$$

and use of incompressibility has been made.

We note that this expansion does not converge uniformly in $\boldsymbol{x}$, and the diffusion coefficient (A7) may exhibit negative values, thereby introducing unphysical singularities into the problem. In other words, Eqs. (A6) and (A7) do not constitute a truly Markovian process with well-defined corresponding Langevin equation driven by white noise. In general these equations are valid only for $\tau \ll 1$ and $\tau/\sigma^2 \ll 1$ (in dimensionless units).

APPENDIX B: GENERALIZED FORMULAS IN THE STEADY SHEAR FLOW CASE

The exact expression of the eddy diffusivity tensor for a stationary bidimensional shear flow,

$$\boldsymbol{v} = (v(y),0), \quad \text{(B1)}$$

with $v(y)$ a periodic function in y can be deduced from Eq. (34) and reads

$$D_{11}^E = \sigma^2 + \frac{1}{2\pi}\int dk \frac{|\hat{v}(k)|^2}{\sigma^2 k^2} + \tau \frac{1}{2\pi}\int dk |\hat{v}(k)|^2 + O(\tau^2); \quad D_{12}^E = 0; \quad D_{22}^E = \sigma^2. \quad \text{(B2)}$$

Applying the generalized formulas (65), (66), and (67) to the shear flow we want to obtain the expressions (B2).

For the shear flow Eq. (65) reads

$$(\partial_t + v\partial_1 - \sigma^2\partial^2) w_1^{(0)} = -v, \quad \text{(B3)}$$

$$(\partial_t + v\partial_1 - \sigma^2\partial^2) w_2^{(0)} = 0. \quad \text{(B4)}$$

Taking the Fourier transform $\hat{v}(k)$ of $v(y)$ we obtain the stationary solution $w_2^{(0)}=0$ and

$$w_1^{(0)}(y) = -\frac{1}{2\pi}\int dk \frac{\hat{v}(k)}{\sigma^2 k^2} e^{iky}. \quad \text{(B5)}$$

Thus Eq. (66) becomes

$$(\partial_t + v\partial_1 - \sigma^2\partial^2) w_1^{(1)} = (p-q)\sigma^2 \partial_2^2 v, \quad \text{(B6)}$$

$$(\partial_t + v\partial_1 - \sigma^2\partial^2) w_2^{(1)} = 0. \quad \text{(B7)}$$

The stationary solutions are

$$w_1^{(1)}(y) = -(p-q)v(y); \quad w_2^{(1)} = 0. \quad \text{(B8)}$$

Using Eq. (67) the eddy diffusivity tensor is

$$D_{12}^E = 0, \quad \text{(B9)}$$

$$D_{22}^E = \sigma^2, \quad \text{(B10)}$$

$$D_{11}^E = \sigma^2 - \frac{1}{2}\langle v w_1^{(0)} \rangle - \frac{\tau}{2}(q\sigma^2 \langle v \partial_2^2 w_1^{(0)} \rangle + \langle v w_1^{(1)} \rangle) + O(\tau^2) \quad \text{(B11)}$$

and

$$D_{11}^E = \sigma^2 + \frac{1}{2\pi}\int dk \frac{|\hat{v}(k)|^2}{\sigma^2 k^2} + \tau(p-2q)\frac{1}{2\pi}\int dk |\hat{v}(k)|^2 + O(\tau^2). \quad \text{(B12)}$$

A comparison between Eqs. (B12) and (B2) shows that the generalized equations lead to the exact expression of the eddy diffusivity tensor for a stationary shear flow if

$$p = 2q + 1.$$

The same condition is found if we consider a time-dependent shear flow.

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